SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2006
YEAR 12
ASSESSMENT TASK #2

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.
- Hand in your answer booklets in 3 sections. Section A (Question 1), Section B (Question 2) and Section C (Question 3)

Total Marks – 74

- Attempt questions 1-3
- All sections are not of equal value

Examiner: *R.Boros*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a} \sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a} \cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a} \tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2} \right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Section A – Start a new booklet

Marks

Question 1. (22 marks)

a) Differentiate the following expressing your answer in its simplest form:

(i) $y = \ln(x+1)$ 2

(ii) $y = \sin^{-1} 2x$ 2

b) Evaluate (leaving your answer in exact form).

(i) $\displaystyle\int_0^2 \frac{t}{t^2+1}\,dt$ 2

(ii) $\displaystyle\int_0^1 \frac{dx}{\sqrt{2-x^2}}$ 2

c) Find the equation (written in general form) of the normal to the curve $y = \tan^{-1} 2x$, at the point where $x = \frac{1}{2}$. 3

d) Find the equation (written in general form) of the tangent to $y = e^{\tan^{-1} x}$ at the point where the curve cuts the *y* axis. 3

e) (i) Differentiate $x\tan^{-1} 3x$ with respect to *x*. 4

(ii) Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{1}{3}} \left(\tan^{-1} 3x + \frac{3x}{1+9x^2} \right) dx$

f) The function $y = e^{-kx}$ satisfies $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = 0$ 4

(i) Show that $k^2 - 4k + 3 = 0$

(ii) Hence, find the possible values of *k*.

End of Section A

Section B – Start a new booklet

Question 2. (25 marks)

a) Find $\lim_{x \to 0} \dfrac{\sin 2x}{x}$. 1

b) Evaluate $\cos\left(\sin^{-1}\left(-\dfrac{1}{2}\right)\right)$, leaving your answer in exact form. 1

c) Without using a calculator, show that: 3

$$\tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{4}\right) = \tan^{-1}\left(\frac{2}{9}\right)$$

d) Solve the equation: 3

$$\ln\left(x^3 + 19\right) = 3\ln\left(x+1\right).$$

e) Taking $x = 3$ as the first approximation to the root of $x^2 - \ln x - 10 = 0$, use Newtons' Method once to find another approximation correct to two decimal places. 3

f) What is the domain of the function $f(x) = \dfrac{x}{\ln(x-1)}$. 2

g) Consider the function $y = 2\sin^{-1}\left(\dfrac{x}{3}\right)$. 5

(i) State the domain.

(ii) State the range.

(iii) Hence, or otherwise, sketch the function.

Question 2 continues overleaf.

h) A function is given by the rule $f(x) = \frac{x+1}{x+2}$. 2

Find the rule for the inverse function $f^{-1}(x)$.

i) The diagram below shows the graph of the function $y = xe^{-x}$. 5

A is a stationary point on the curve.

(i) Show that A is the point $\left(1, \frac{1}{e}\right)$.

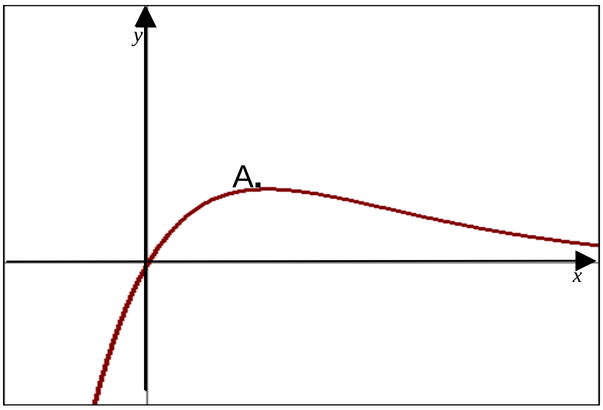
(ii) State the range of the function $y = xe^{-x}$

(iii)How many real roots are there to the equation $xe^{-x} = k$ if

(a.) $0 < k < \frac{1}{e}$

(b.) $k \le 0$

(c.) $k > \frac{1}{e}$



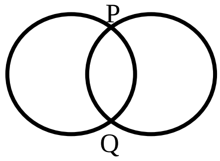
End of Section B

Section C – Start a new booklet

Question 3. (27 marks)

a) A cosine curve has an amplitude of 5 and a period of 3π. It has a minimum turning point at $(0,5)$. Find its equation. 3

b) 6



Two circles each with radius 2cm intersect at P and Q. The common chord PQ subtends an angle θ radians at each centre.

(i) Show that the area $A cm^2$ of the overlapping part of the circles in given by $A = 4\theta - 4\sin\theta$

(ii) If the three regions shown in the diagram all have the same area, show that $\theta - \sin\theta - \frac{\pi}{2} = 0$

c) The area between the curve $y = \sin^2 x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$, is rotated through one complete revolution about the x-axis. 5

(i) Find the exact value of the **area** involved.

(ii) Use Simpson's Rule with 3 function values to find an approximation to the volume of the solid of revolution leaving your answer in terms of π.

Question 3 continues overleaf.

d) Given $f(x) = \dfrac{8}{4+x^2}$. 8

(i) Show that $f(x)$ is an even function.

(ii) Sketch a graph of $y = f(x)$.

(iii) The line $y = 1$ meets the curve at 2 points P and Q. Determine the x coordinates of P and Q.

(iv) Calculate the exact area of the region enclosed by the interval PQ and the arc PQ of the curve.

(v) The region in (iv) makes a revolution about the *y*-axis, show that the volume of the solid formed is $4\pi\left(2\ln 2 - 1\right)\ units^3$.

e) (i) By using the Principle of Mathematical Induction, prove that: 5

$$6\left(1^2+2^2+3^2+\ldots+n^2\right) = n\left(n+1\right)\left(2n+1\right)$$

(ii) Hence, find the value of the limit:

$$\lim_{n\to\infty}\left(\frac{1^2+2^2+3^2+\ldots+n^2}{n^3}\right)$$

End of Examination

Question 1.

a) i) $\frac{dy}{dx} = \frac{1}{x+1}$

ii) $y = \sin^{-1}\frac{x}{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{(\frac{1}{2})^2 - x^2}}$$

$$= \frac{1}{\sqrt{\frac{1}{4} - \frac{4x^2}{4}}}$$

$$= \frac{2}{\sqrt{1-4x^2}}$$

b) i) $\int_0^2 \frac{t}{t^2+1}\,dt = \frac{1}{2}\int_0^2 \frac{2t}{t^2+1}\,dt$

$$= \frac{1}{2}\left[\ln(t^2+1)\right]_0^2$$

$$= \frac{1}{2}(\ln 5 - \ln 1)$$

$$= \frac{1}{2}\ln 5$$

ii) $\int_0^1 \frac{dx}{\sqrt{2-x^2}} = \int_0^1 \frac{1}{\sqrt{\sqrt{2}^2 - x^2}}\,dx$

$$= \left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1$$

$$= \sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1} 0 = \frac{\pi}{4}.$$

~~4~~ 2

c) Normal to $y = \tan^{-1} 2x$ at $x = \frac{1}{2}$

$$y = \frac{1}{2}\left(2\tan^{-1}\frac{x}{\frac{1}{2}}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{(\frac{1}{2})^2 + x^2}\right)$$

$$= \frac{1}{2} \times \frac{1}{\frac{1}{4} + \frac{4x^2}{4}}$$

$$= \frac{2}{1+4x^2}$$

Grad at $x = \frac{1}{2}$.

$$m = \frac{2}{1+4(\frac{1}{2})^2}$$

$$= 1$$

$$m_\perp = -1.$$

Point on curve at $x = \frac{1}{2}$.

$$y = \tan^{-1}(1).$$

$$= \frac{\pi}{4}$$

$$\left(\frac{1}{2}, \frac{\pi}{4}\right).$$

Eqn of normal.

$$y - \frac{\pi}{4} = -\left(x - \frac{1}{2}\right)$$

$$y - \frac{\pi}{4} = -x + \frac{1}{2}.$$

$$x + y - \frac{\pi}{4} - \frac{1}{2} = 0$$

$$4x + 4y - \pi - 2 = 0.$$

Tangent to

d) $y = e^{\tan^{-1}x}$ at $x = 0$.

$$\frac{dy}{dx} = e^{\tan^{-1}x}\left(\frac{1}{1+x^2}\right)$$

Grad at $x = 0$.

$$m = e^{\tan^{-1}0}\left(\frac{1}{1+0^2}\right)$$

$$= e^0$$
$$= 1.$$

Point on curve at $x=0$

$$y = e^{\tan^{-1}0}$$

$$= 1$$

$(0, 1)$.

eqn of tangent.

$$y - 1 = x$$

$$x - y + 1 = 0.$$

e) i) $f(x) = x\tan^{-1}3x$.

$$= \frac{1}{3}x \times 3\tan^{-1}\frac{x}{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3} \times 3\tan^{-1}\frac{x}{\frac{1}{3}} + \frac{1}{3}x\frac{1}{(\frac{1}{3})^2 + x^2}$$

$$= \tan^{-1}3x + \frac{1}{3} \times \frac{9x}{1+9x^2}$$

$$= \tan^{-1}3x + \frac{3x}{1+9x^2}$$

ii) $\int_0^1 \tan^{-1}3x + \frac{3x}{1+9x^2}\,dx = \left[x\tan^{-1}3x\right]_0^{\frac{1}{3}}$

$$= \frac{1}{3}\tan^{-1}1 - 0$$

$$= \frac{\pi}{12}.$$

f) i) $y = e^{-kx}$

$$\frac{dy}{dx} = -ke^{-kx}$$

$$\frac{d^2y}{dx^2} = k^2e^{-kx}.$$

So $k^2e^{-kx} - 4ke^{-kx} + 3e^{-kx} = 0$.

Since $e^{-kx} \neq 0 \ \forall x$ we can divide through by e^{-kx}

$\therefore k^2 - 4k + 3 = 0$.

f) ii) $k^2 - 4k + 3 = 0$

$(k-3)(k-1) = 0$

$k = 3, 1.$

QUESTION 2

(a) $\lim_{x\to 0} \frac{\sin 2x}{x} = 2\lim_{x\to 0} \frac{\sin 2x}{2x}$

$= 2$

(b) $\cos\left[\sin^{-1}(-\tfrac{1}{2})\right] = \cos(-\tfrac{\pi}{6})$

$= \sqrt{3}/2$

(c) Let $x = \tan^{-1}(\tfrac{1}{2})$ + $y = \tan^{-1}(\tfrac{1}{4})$

$\tan x = \tfrac{1}{2}$ $\quad \tan y = \tfrac{1}{4}$

$\tan(x-y) = \dfrac{\tfrac{1}{2} - \tfrac{1}{4}}{1 + \tfrac{1}{2}\cdot\tfrac{1}{4}}$

$= \tfrac{2}{9}$

$x - y = \tan^{-1}(\tfrac{2}{9})$

$\tan^{-1}(\tfrac{1}{2}) - \tan^{-1}(\tfrac{1}{4}) = \tan^{-1}(\tfrac{2}{9})$

(d) $\ln(x^3+19) = 3\ln(x+1)$

$x^3 + 19 = (x+1)^3$

$x^3 + 19 = x^3 + 3x^2 + 3x + 1$

$3x^2 + 3x - 18 = 0$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = 2 \quad (x > -1)$

e $f(x) = x^2 - \ln x - 10$

$f'(x) = 2x - \tfrac{1}{x}$

$a_1 = 3 - \dfrac{3^2 - \ln 3 - 10}{6 - \tfrac{1}{3}}$

approx $x \approx 3.37$

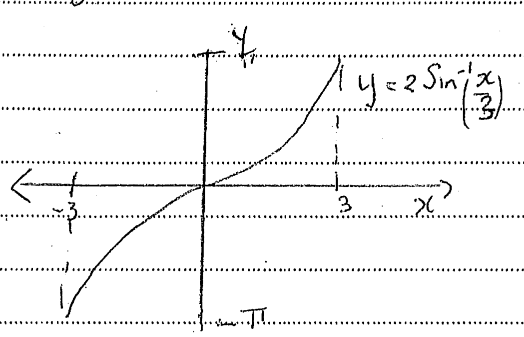
(f) $x - 1 > 0 \quad x > 1$

$\ln(x-1) \neq 0 \quad x - 1 \neq 1$

$x > 1$ + $x \neq 2$

(g) Domain $-3 \leq x \leq 3$

Range $-\pi \leq y \leq \pi$



(h) $f(x) = \dfrac{x+1}{x+2} = y$

inverse $x = \dfrac{y+1}{y+2}$

$xy + 2x = y + 1$

$y(x-1) = 1 - 2x$

$f^{-1}(x) = y = \dfrac{1-2x}{x-1}$

(i) $y = xe^{-x}$

(i) $y' = e^{-x}.1 + x.-e^{-x}$

$= e^{-x}(1-x)$

$y' = 0 \quad x = 1 \quad y = \tfrac{1}{e}$

only one solution

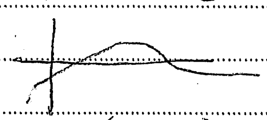
It is the point $(1, \tfrac{1}{e})$

(ii) range $y \leq 1/e$

(iii) $xe^{-x} = k$

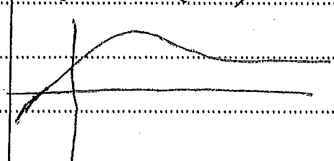
$xe^{-x} - k = 0$

(i) curve moves down less than $\tfrac{1}{e}$ units



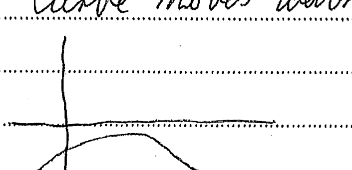
2 roots

(ii) $xe^{-x} - (-k)$ curve moves up



1 root

(iii) Curve moves down more than $\tfrac{1}{e}$ u



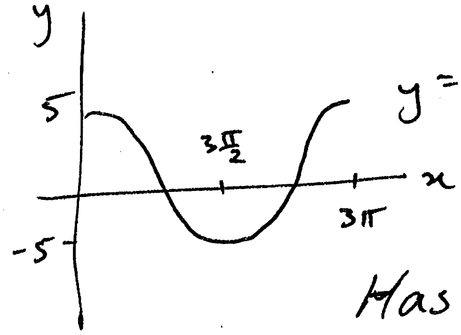
no roots.

Question 3

(a) period $= 3\pi = \frac{2\pi}{b}$

$\therefore b = \frac{2}{3}$

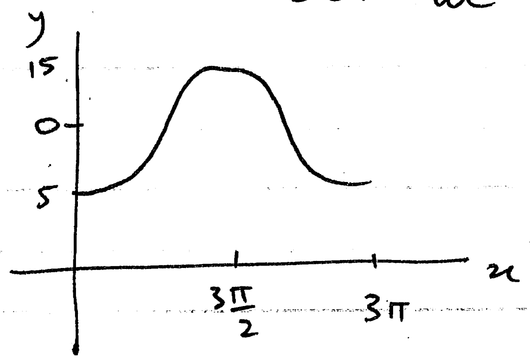
$a = 5$



$y = 5\cos\left(\frac{2}{3}x\right)$

Has the right shape

BUT we want.



The curve we want shifts $y = 5\cos\left(\frac{2}{3}x\right)$

horizontally $\frac{3\pi}{2}$ units (to left or right)

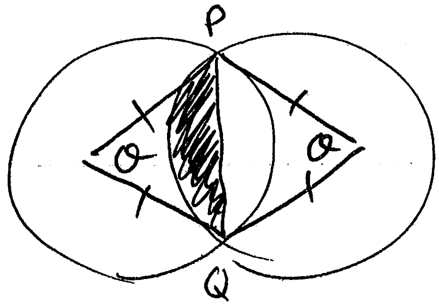
vertically 10 units (up)

$\therefore y = 5\cos\left(\frac{2}{3}\left(x + \frac{3\pi}{2}\right)\right) + 10$ OR $y = 5\cos\left(\frac{2}{3}\left(x - \frac{3\pi}{2}\right)\right) + 10$

$y = 5\cos\left(\frac{2}{3}x + \pi\right) + 10$ $\quad y = 5\cos\left(\frac{2}{3}x - \pi\right) + 10$

OR $y = -5\cos\left(\frac{2}{3}x\right) + 10$

(b)



(i) Area of minor segment (shaded) $= \frac{1}{2}r^2(\theta - \sin\theta)$

$$= \frac{1}{2}(2)^2(\theta - \sin\theta)$$
$$= 2(\theta - \sin\theta)$$

Area concerned

$$A = 2(2(\theta - \sin\theta))$$
$$A = 4(\theta - \sin\theta)$$
$$A = 4\theta - \sin\theta)$$

(ii) Area of circle $= \pi r^2$

$$= \pi(2)^2$$
$$= 4\pi$$

If the three regions have the same area

Area of circle $= 2A$

$$4\pi = 2(4\theta - 4\sin\theta)$$
$$4\pi = 8\theta - 8\sin\theta$$

$$8\theta - 8\sin\theta - 4\pi = 0$$
$$\theta - \sin\theta - \frac{\pi}{2} = 0$$

(c)(i) $A = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2x)\, dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\frac{\pi}{2} - \cancel{\frac{1}{2}\sin\pi} - \left(0 - \cancel{\frac{1}{2}\sin 0}\right)\right]$

$= \frac{\pi}{4}$ units^2

(ii) $V = \pi \int_0^{\frac{\pi}{2}} y^2 \, dx$

$= \pi \int_0^{\frac{\pi}{2}} \sin^4 x \, dx$

$f(x) = \sin^4 x$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$f(x)$	0	$\frac{1}{4}$	1

$\int_0^{\frac{\pi}{2}} \sin^4 x \, dx \approx \frac{\frac{\pi}{2}}{6}\left[0 + 4\left(\frac{1}{4}\right) + 1\right]$

$= \frac{\pi}{12}[2]$

$= \frac{\pi}{6}$

$\therefore V = \pi \int_0^{\frac{\pi}{2}} \sin^4 x \, dx$

$\approx \frac{\pi^2}{6}$ units^3

(d)(i) $f(x) = \dfrac{8}{4+x^2}$

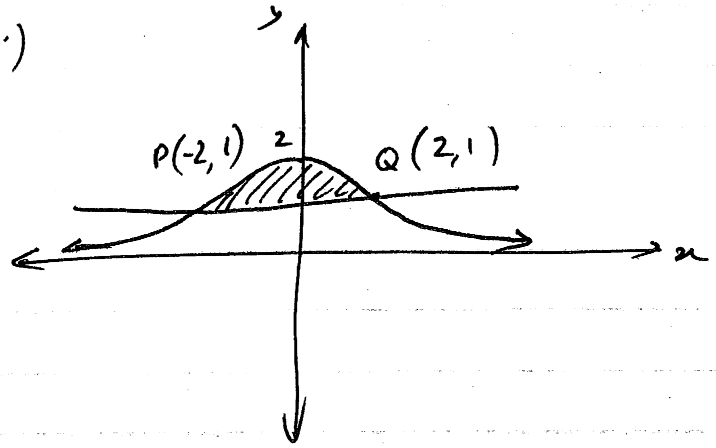
$f(-x) = \dfrac{8}{4+(-x)^2}$

$= \dfrac{8}{4+x^2}$

since $f(x) = f(-x)$

$f(x)$ is even

(ii)



(iii) P has x coordinate -2

Q has x coordinate 2

(iv) Area $= 2\left[\int_0^2 \dfrac{8}{4+x^2}\,dx - 2\times 1\right]$

$= 2\left[8\left[\tfrac{1}{2}\tan^{-1}\tfrac{x}{2}\right]_0^2 - 2\right]$

$= 2\left[4\tan^{-1}1 - \cancel{4\tan^{-1}0} - 2\right]$

$= 2\left[4\cdot\tfrac{\pi}{4} - 2\right]$

$= 2\left[\pi - 2\right]$ units^2

(v) $y = \dfrac{8}{4+x^2}$

$4 + x^2 = \dfrac{8}{y}$

$x^2 = \dfrac{8}{y} - 4$

$V = \pi \int_1^2 x^2 \, dy$

$V = \pi \int_1^2 \left(\dfrac{8}{y} - 4 \right) dy$

$= \pi \left[8 \ln y - 4y \right]_1^2$

$= \pi \left[8 \ln 2 - 4(2) - (\cancel{8 \ln 1} - 4(1)) \right]$

$= \pi \left[8 \ln 2 - 4 \right]$

$= 4\pi (2 \ln 2 - 1)$

(e) $6(1^2 + 2^2 + 3^2 + \dots + n^2) = n(n+1)(2n+1)$

Prove true for $n = 1$

$LHS = 6(1)^2$
$= 6$

$RHS = (1)(1+1)(2(1)+1)$
$= 1.2.3$
$= 6$

$LHS = RHS$

$\therefore$ true for $n = 1$

Assume true for $n = k$ where k is a positive integer.

$6(1^2 + 2^2 + 3^2 + \dots + k^2) = k(k+1)(2k+1)$

Prove true for $n=k+1$

<u>ie</u> Prove $6(1^2+2^2+\dots+k^2+(k+1)^2) = (k+1)(k+2)(2(k+1)+1)$
$= (k+1)(k+2)(2k+3)$

$LHS = 6(1^2+2^2+\dots+k^2) + 6(k+1)^2$

using assumption.

$= k(k+1)(2k+1) + 6(k+1)^2$

$= (k+1)(2k^2+k+6k+6)$

$= (k+1)(2k^2+7k+6)$

$= (k+1)\underline{(2k+3)(\cancel{2}^{1}k+\cancel{4}^{2})}$

$= (k+1)(k+2)(2k+3)$

$= RHS$

$\times\ 12$
$+\ 7$
$3, 4$

$\therefore$ true for $n=k+1$

If true for $n=k$ it is true for $n=k+1$. Since true for $n=1$ by the principle of mathematical induction it is true for all positive integers n.

(ii) $\lim\limits_{n\to\infty} \dfrac{(1^2+2^2+3^2+\dots+n^2)}{n^3}$

$= \lim\limits_{n\to\infty} \dfrac{n(n+1)(2n+1)}{6n^3}$

$= \lim\limits_{n\to\infty} \dfrac{2n^3+3n^2+n}{6n^3}$

$= \lim\limits_{n\to\infty} \dfrac{\cancel{\frac{2n^3}{n^3}} + \frac{3n^2}{n^3} + \frac{n}{n^3}}{\cancel{\frac{6n^3}{n^3}}} = \dfrac{1}{3}$